technical proof

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TP B.29

Simulation of a CB striking two frozen OBs along their line of centers

supporting:

"The Illustrated Principles of Pool and Billiards"

http://billiards.colostate.edu

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originally posted: 8/6/2024 last revision: 8/7/2024

From "Impact Mechanics" by Stronge (2000) on p. 118, based on Hertz contact stress calculations, for two balls of radius "R" colliding, the radius of the contact patch "a" for a given compressive force "F" between the balls is given by:

$$a = \frac{R}{2} \cdot (c \cdot F)^{\frac{1}{3}}$$

where "c" is a constant related to ball properties.

The force is related to the total compression of the balls "δ" according to:

$$F = \frac{1}{c} \cdot \left(\frac{2 \cdot \delta}{R}\right)^{\frac{3}{2}}$$

The constant "c" can be found from this equation if the force and compression are known:

$$c = \frac{\left(\frac{2 \cdot \delta}{R}\right)^{\frac{3}{2}}}{F}$$

Here are numbers from experimental data reported by Marlow on p. 45 of "The Physics of Pocket Billiards" (1995):

$R := 1.125 \cdot in$ ball radius

$M := 6 \cdot oz$ ball mass

$v := 1 \cdot \frac{m}{s} = 2.237 \cdot mph$ ball speed

$\Delta T := 284 \cdot 10^{-6} \cdot s$ ball contact time

$\varepsilon := 3.49 \cdot 10^{-3}$ ball strain (ΔR/R) at full compression

$\delta := 2 \cdot R \cdot \varepsilon = 0.2 \cdot mm$ total compression of both balls at full compression

$F := 2270 \cdot N = 510.3 \cdot lbf$ force at full compression

The constant "c" in the equation above can be calculated from this experimental data:

$$c := \frac{\left(\frac{2\cdot\delta}{R}\right)^{\frac{3}{2}}}{F} = 7.266\times 10^{-7}\,\frac{1}{N}$$

The contact patch size can be calculated using this value:

$$a := \frac{R}{2}\cdot(c\cdot F)^{\frac{1}{3}} = 1.69\cdot mm \qquad 2\cdot a = 3.38\cdot mm$$

Stripping units off everything (assuming SI units) for the program below to work properly:

$$R := \frac{R}{m} = 0.029 \qquad M := \frac{M}{kg} = 0.17 \qquad v := \frac{v}{\frac{m}{s}} = 1$$

$$\Delta T := \frac{\Delta T}{s} = 2.84\times 10^{-4} \qquad c := c\cdot N = 7.266\times 10^{-7}$$

Knowing the constant "c," the compressive force between two balls can be calculated, given the amount of compression "δ" between the balls:

$$F(\delta) := \frac{1}{c}\cdot\left(\frac{2\cdot\delta}{R}\right)^{\frac{3}{2}}$$

Simulation of a CB striking two frozen balls along the line of centers

Now a simple numerical simulation will be performed to see how three balls move during the collision, where the CB strikes two frozen OBs along their line of centers. The CB is labeled as ball "1" and the OBs are labeled "2" and "3." The position, speed, and acceleration of each ball are designated by "x," "v," and "a." The force between the first two balls is "F_{12}," and the force between the 2nd and 3rd balls is "F_{23}." A more proper Runge Kutta approach, numerically solving the simultaneous non-linear equations of motion would be more efficient, but with a small enough time step, the simple procedure below is sufficient, where the speed and acceleration are assumed to be constant during each time step:

$$true := 1 \qquad n := 10000 \qquad \Delta t := \frac{\Delta T}{n} \qquad D := 2\cdot R$$

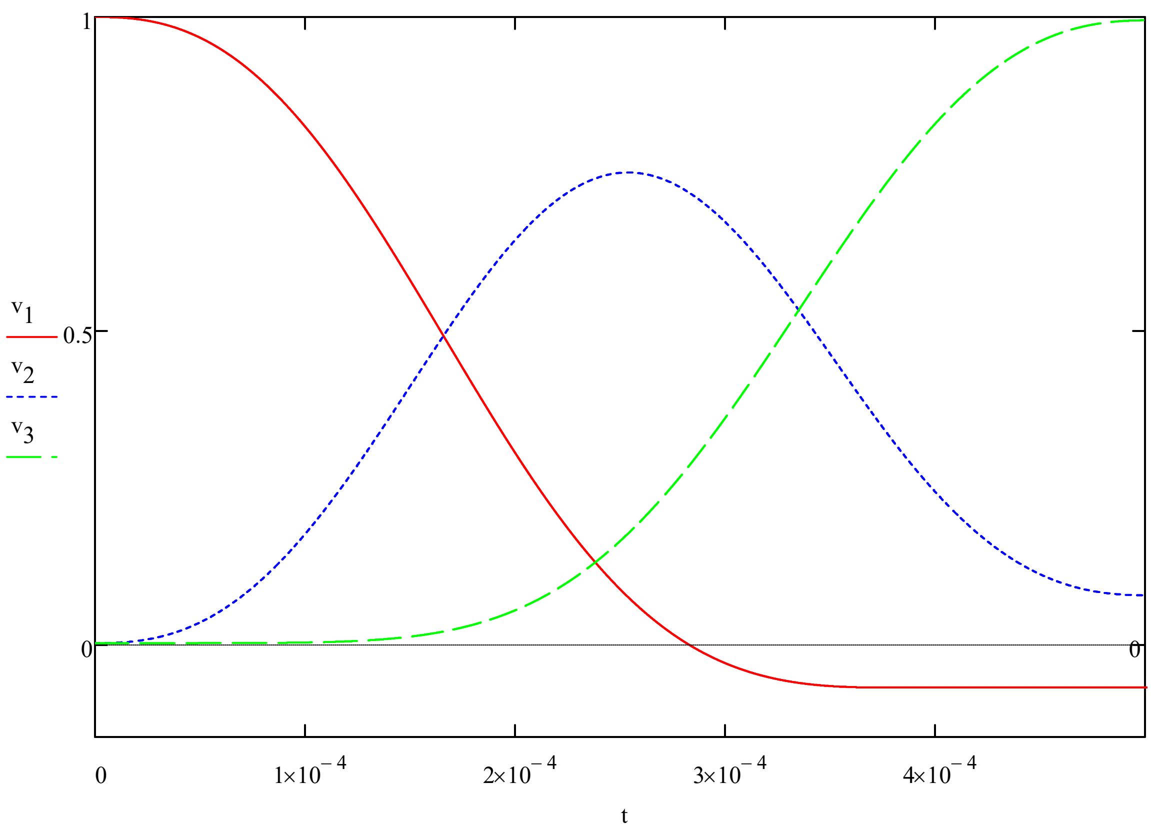
$$\begin{pmatrix} n_t \\ v_1 \\ v_2 \\ v_3 \\ F_{12} \\ F_{23} \end{pmatrix} := \left| \begin{array}{l} i \leftarrow 0 \\ x_{1_0} \leftarrow 0 \\ x_{2_0} \leftarrow D \\ x_{3_0} \leftarrow 2 \cdot D \\ v_{1_0} \leftarrow v \\ v_{2_0} \leftarrow 0 \\ v_{3_0} \leftarrow 0 \\ a_{1_0} \leftarrow 0 \\ a_{2_0} \leftarrow 0 \\ a_{3_0} \leftarrow 0 \\ F_{12_0} \leftarrow 0 \\ F_{23_0} \leftarrow 0 \\ \text{while (true)} \\ \quad \left| \begin{array}{l} x_{1_{i+1}} \leftarrow x_{1_i} + v_{1_i} \cdot \Delta t + \frac{1}{2} \cdot a_{1_i} \cdot \Delta t^2 \\ x_{2_{i+1}} \leftarrow x_{2_i} + v_{2_i} \cdot \Delta t + \frac{1}{2} \cdot a_{2_i} \cdot \Delta t^2 \\ x_{3_{i+1}} \leftarrow x_{3_i} + v_{3_i} \cdot \Delta t + \frac{1}{2} \cdot a_{3_i} \cdot \Delta t^2 \\ v_{1_{i+1}} \leftarrow v_{1_i} + a_{1_i} \cdot \Delta t \\ v_{2_{i+1}} \leftarrow v_{2_i} + a_{2_i} \cdot \Delta t \\ v_{3_{i+1}} \leftarrow v_{3_i} + a_{3_i} \cdot \Delta t \\ \delta_{12} \leftarrow D - \left(x_{2_{i+1}} - x_{1_{i+1}} \right) \\ \delta_{23} \leftarrow D - \left(x_{3_{i+1}} - x_{2_{i+1}} \right) \\ \text{break if } \left[\left(\delta_{12} \le 0 \right) \wedge \left(\delta_{23} \le 0 \right) \right] \\ F_{12_{i+1}} \leftarrow \text{if}\left(\delta_{12} > 0, F\left(\delta_{12}\right), 0 \right) \\ F_{23_{i+1}} \leftarrow \text{if}\left(\delta_{23} > 0, F\left(\delta_{23}\right), 0 \right) \\ a_{1_{i+1}} \leftarrow \dfrac{-F_{12_{i+1}}}{M} \\ a_{2_{i+1}} \leftarrow \dfrac{\left(F_{12_{i+1}} - F_{23_{i+1}} \right)}{M} \end{array} \right. \end{array} \right.$$

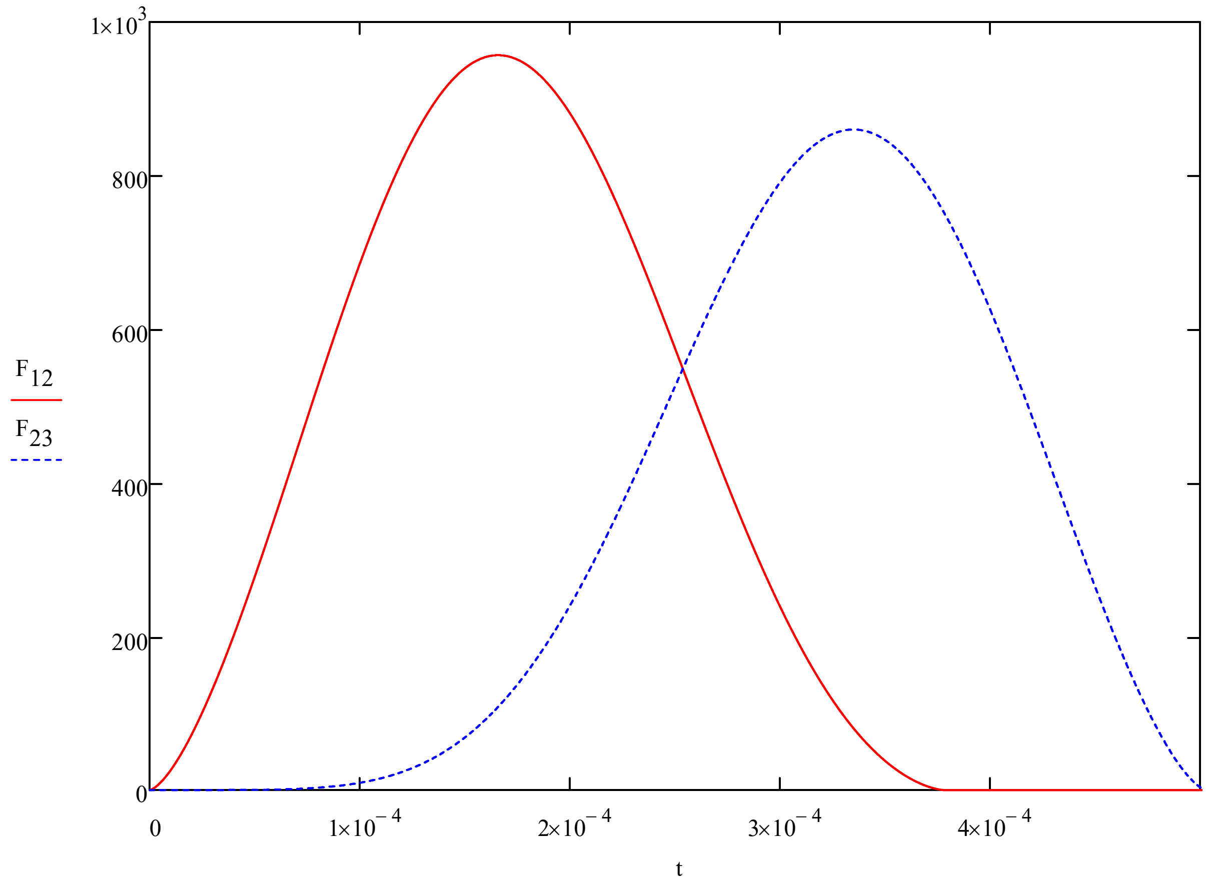
$$a_{3_{i+1}} \leftarrow \frac{F_{23_{i+1}}}{M}$$

$$i \leftarrow i + 1$$

$$\text{return } \left(i + 1 \quad v_1 \quad v_2 \quad v_3 \quad F_{12} \quad F_{23}\right)^T$$

$n_t = 1.772 \times 10^4$ $\qquad$ $i := 0 .. n_t$ $\qquad$ $t_i := i \cdot \Delta t$





starting ball 1 speed:

$$v1_0 = 1$$

ending ball speeds:

$$v1_{n_t} = -0.071 \qquad v2_{n_t} = 0.076 \qquad v3_{n_t} = 0.995$$

momentum is conserved:

$$v1_{n_t} + v2_{n_t} + v3_{n_t} = 1$$

energy is conserved:

$$\left(v1_{n_t}\right)^2 + \left(v2_{n_t}\right)^2 + \left(v3_{n_t}\right)^2 = 1$$

collision times:

two balls: $\Delta T = 2.84 \times 10^{-4}$ three balls: $n_t \cdot \Delta t = 5.031 \times 10^{-4}$

$$\frac{n_t \cdot \Delta t}{\Delta T} = 1.771$$

Results of the simulation:

- The CB bounces back from the two OBs slightly, and separates from the OBs while they are still interacting.
- The 1st OB moves forward slightly after the collision.
- The 2nd OB has a similar speed after the collision as the CB did before the collision.
- The CB feels a larger force since it is pushing against two frozen balls. The 2nd OB feels less force since it has no ball to its right and since the CB separates while the OBs are still interacting.
- During compression, all 3 balls are moving forward together, partly explaining why the 1st OB moves forward after the collision. Also, the force from the CB side is larger, also explaining the forward motion of the 1st OB.
- The total collision time for the 3 balls is 77% longer than the collision time for 2 balls.